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A UNIVERSAL SECOND-PLAYER STRATEGY IN A LINEAR DIFFERENTIAL GAME*

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A two-player zero-sum linear differential game with a fixed termination time, a convex terminal payoff function, and geometrical constraints on player controls is considered. In a previous study, the optimal strategy of the second player (the payoff maximizer) who uses minimum information on the value function of the game was implemented as a piecewise-programmed control, which required specification of certain parameters dependent on the initial position /1/. The present study proposes a second-player strategy in which the control is synthesized by the feedback principle using switching surfaces. The strategy may become non-optimal only in case of sliding on the switching surfaces. This construction was considered in /2/ for the first player with scalar control.

1. ***Statement of the problem.*** Assume that the system dynamics is described by the relationships

$$y^{\cdot} = B(t)u + C(t)v, \quad y \in R^n, \quad u \in P, \quad v \in Q \tag{1.1}$$

Here u and v are the vector control parameters of the first and second player, and P and Q are convex compact sets. The control process terminates at a given time ϑ. The performance criterion is the value of the terminal function $\gamma(y(\vartheta))$ in state $y(\vartheta)$. The first player minimizes and the second player maximizes the value of $\gamma(y(\vartheta))$. It is required to construct an optimal strategy of the second player in the game (1.1).

Let Γ be the value function of the game (1.1). Fix the interval $T = [t_{00}, \vartheta]$, the number c^*, and the non-increasing function $c_*(t)$: $c^* > c_*(t) > \min_x \Gamma(t, x)$. We assume that the set $\Omega = \{(t, x) \in T \times R^n: c_*(t) \leqslant \Gamma(t, x) \leqslant c^*\}$ contains the region of relevant initial positions. Let $W_c = \{(t, x) \in T \times R^n: \Gamma(t, x) \leqslant c\}$ be the level set of the function Γ and $W_c(t) = \{x \in R^n: (t, x) \in W_c\}$ its section at time t. We know that the function $\Gamma(t, \cdot)$ is convex for all $t \leqslant \vartheta$. Therefore, the sets $W_c(t)$ are also convex. We will additionally assume that for all $t \in T$, $c_*(t) \leqslant c \leqslant c^*$, the sets $W_c(t)$ are bounded and have a non-empty interior.

2. ***General specification of the second-player optimal strategy.*** We will now describe the second optimal strategy, whose existence and form were established in /3/ **. (**See also Zarkh M.A., A universal second-player optimal strategy in a linear differential game, Sverdlovsk, 1985. Unpublished manuscript, VINITI 25.10.85, 7438-V85.) Below we will define its generalization.

We will introduce the necessary notation. For $l \in R^n$ and $t \leqslant \vartheta$, let

$$Q(l, t) = \{\bar{q} \in Q: l'C(t)\bar{q} = \max l'C(t)q, q \in Q\}$$

The symbol $\rho(\cdot, W)$ denotes the support function. With each position $(t, x) \in \Omega$ we associate the cone

$$K(t, x) = \{l \in R^n: l'x = \rho(l, W_{\Gamma(t,x)}(t))\}$$

Alternatively, the cone $K(t, x)$ can be defined as the conical hull of the subdifferential of the function $\Gamma(t, \cdot)$ at the point x. Note that $K(t, x)$ is a convex closed cone that is not zero and does not contain linear subspaces. Let $L(t, x)$ be the collection of extreme unit vectors of the cone $K(t, x)$. An extreme vector /4/ is a vector which cannot be represented as the sum of two linearly independent vectors from the cone.

Define the second-player strategy

$$V^\circ(t, x) = \bigcup_{l \in L(t, x)} Q(l, t), (t, x) \in \Omega; \quad V^\circ(t, x) = Q, (t, x) \notin \Omega$$

The strategy V° is optimal /3/ for any initial position from Ω. The definition of the strategy outside Ω is purely formal and irrelevant.

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Define a new strategy

$$\bar{V}^{\circ}(t, x) = \bigcup_{l \in \bar{L}(t, x)} Q(l, t)$$

where

$$\bar{L}(t, x) = \{l = \lim l_k\colon\ l_k \in L(t_k, x_k),\ (t, x) = \lim (t_k, x_k)\}$$

Let us list the main properties* (*The proofs are given in Zarkh M.A., Second-player positional control in a linear differential game, Moscow, 1989. Unpublished manuscript, VINITI No.6840-V89.) of the strategy $\bar{V}^{\circ}$: the strategy $\bar{V}^{\circ}$ is optimal in Ω, the inclusion $V^{\circ}(t, x) \subset \bar{V}^{\circ}(t, x)$ holds, and the mapping $\bar{V}^{\circ}\colon (t, x) \to \bar{V}^{\circ}(t, x)$ is upper semicontinuous by inclusion.

Note that the strategy $\bar{V}^{\circ}$ is many-valued, i.e., the optimal second-player control in position (t, x) in general is not unique and may be chosen from the subset $\bar{V}^{\circ}(t, x)$ of the compactum Q. If V^* is some strategy that in each position (t, x) satisfies the inclusion $V^*(t, x) \subset \bar{V}^{\circ}(t, x)$, then it is optimal in Ω.

3. ***Switching surfaces.*** Let us consider some special cases in which the strategy V^* is constructed in a computationally convenient form using switching surfaces. Assume that the set $Q \subset R^s$ is a parallelepiped of the form

$$Q = \{q = (q_1, \ldots, q_s) \in R^s\colon\ |q_i| \leqslant \nu_i,\quad i = 1, 2, \ldots, s\}$$

Let e_i be the i-th unit vector of the space R^s and $h_i(t) = C(t)e_i$ the i-th column of the matrix $C(t),\ i = 1, 2, \ldots, s$. Let

$$\Pi_i(t) = \{x \in R^n\colon\ \Gamma(t, x) = \min \Gamma(t, x + \lambda h_i(t)),\ \lambda \in R^1\}$$
$$D_i^+(t) = \{x \in R^n\colon\ x + \lambda h_i(t) \notin \Pi_i(t),\quad \lambda \geqslant 0\}$$
$$D_i^-(t) = \{x \in R^n\colon\ x + \lambda h_i(t) \notin \Pi_i(t),\ \lambda \leqslant 0\}$$

The set $\Pi_i(t)$ partitions the space R^n into two parts: $D_i^+(t)$ and $D_i^-(t)$. We call $\Pi_i(t)$ a switching surface (SS). The following proposition explains the meaning of this term. Let

$$G_i^+ = \{q \in Q\colon\ q_i = \nu_i\},\quad G_i^- = \{q \in Q\colon\ q_i = -\nu_i\}$$

Proposition 1. Let $(t, x) \in \Omega$. The conditions

$$x \in D_i^+(t)\quad (x \in D_i^-(t)) \tag{3.1}$$

$$l'h_i(t) > 0\ \forall l \in \bar{L}(t, x)\quad (l'h_i(t) < 0\quad \forall l \in \bar{L}(t, x)) \tag{3.2}$$

$$\bar{V}^{\circ}(t, x) \in G_i^+\quad (\bar{V}^{\circ}(t, x) \in G_i^-) \tag{3.3}$$

are equivalent.

Proof. The equivalence of conditions (3.2) and (3.3) follows from the definition of $\bar{V}^{\circ}$ and the representability of the set Q in the form

$$Q = \{\bar{q} + \lambda e_i\colon\ \bar{q} \in G_i,\ -2\nu_i \leqslant \lambda \leqslant 0\}$$

Let us prove the equivalence of (3.1) and (3.2). Assume that (3.1) holds, i.e., $x \in D_i^+(t)$. This means that $x + \lambda h_i(t) \notin \Pi_i(t), \lambda \geqslant 0$. Therefore, $\min_\lambda \Gamma(t, x + \lambda h_i(t)) = \Gamma(t, x + \lambda^* h_i(t))$, where $\lambda^* < 0$, $x + \lambda^* h_i(t) \in \Pi_i(t)$. Since $\Gamma(t, x + \lambda^* h_i(t)) < \Gamma(t, x)$, we have $x + \lambda^* h_i(t) \in \operatorname{int} W_{\Gamma(t, x)}(t)$. Therefore, $l'(x + \lambda^* h_i(t)) < \rho(l, W_{\Gamma(t, x)}(t))$ for all $l \in R^n, l \neq 0$. For $l \in \bar{L}(t, x)$ we have $\rho(l, W_{\Gamma(t, x)}(t)) = l'x$. Thus, $l'(x + \lambda^* h_i(t)) < l'x$. Hence $l'h_i(t) > 0$ for $l \in \bar{L}(t, x)$.

Conversely, assume that (3.2) holds. Since any non-zero vector of a cone can be represented as the sum extreme vectors (Theorem 18.5 /4, p.183/), we have $l' h_i(t) > 0$ for $l \in K(t, x), l \neq 0$. Thus $l'(x + \lambda h_i(t)) < \rho(l, W_{\Gamma(t, x)}(t))$ for $\lambda < 0$ and $l \in K(t, x), l \neq 0$. Therefore, for negative λ close to zero we have $x + \lambda h_i(t) \in \operatorname{int} W_{\Gamma(t, x)}(t)$. Thus, $\min_\lambda \Gamma(t, x + \lambda h_i(t)) = \Gamma(t, x + \lambda^* h_i(t)) < \Gamma(t, x)$, $\lambda^* < 0$. By convexity, the function $\varphi(\lambda) = \Gamma(t, x + \lambda h_i(t))$ increases in the interval $(\lambda^*, +\infty)$. Therefore, $x + \lambda h_i(t) \notin \Pi_i(t)$ for $\lambda \geqslant 0$.

From Proposition 1 we see that in the region $D_i^+(t)\ (D_i^-(t))$ the i-th component of all vectors from the set $\bar{V}^{\circ}(t, x)$ is $\nu_i, (-\nu_i)$.

Define the strategy $V^*(t, x) = (V_1^*(t, x), \ldots, V_s^*(t, x))$ in the form

$$V_i^*(t, x) = \begin{cases} \pm\nu_i, & x \in D_i^{\pm}(t) \\ \{-\nu_i, \nu_i\}, & x \in \Pi_i(t);\ i = 1, 2, \ldots, s \end{cases}$$

Below we state the conditions when $V^*(t, x) \subset \bar{V}^\circ(t, x)$, i.e., when V^* is an optimal strategy. We put

$$G_i(t, x) = \begin{cases} G_i^\pm, & x \in D_i^\pm(t) \\ Q, & x \in \Pi_i(t);\ i = 1, 2, \ldots, s \end{cases}$$

$$G(t, x) = \bigcap_{i=1}^{s} G_i(t, x), \quad I(t, x) = \{i \in 1, 2, \ldots, s: x \in \Pi_i(t)\}$$

Statement 1. $V^*(t, x)$ is the collection of vertices of the face $G(t, x)$.

Statement 2. If $I(t, x) = \varnothing$, then $G(t, x)$ is a one-point set and $G(t, x) = V^*(t, x) = \bar{V}^\circ(t, x)$.

Statement 2 is a direct consequence of the definition of V^* and Proposition 1.

Condition A. For any vertex g of the face $G(t, x)$ there is a sequence of points $\{x_k\}$, such that $x_k \to x$, $I(t, x_k) = \varnothing$, and $G(t, x_k) = g$.

Note that Condition A is satified at all points that do not belong to the SS. In order to verify this, it suffices to take the sequence $x_k \equiv x$ and apply Statement 2.

Statement 3. If Condition A is satisfied at the point (t, x), then

$$V^*(t, x) \subset \bar{V}^\circ(t, x) \tag{3.4}$$

Indeed, by the semicontinuity of $\bar{V}^\circ$ and Statement 2, each vertex of the face $G(t, x)$ is contained in the set $\bar{V}^\circ(t, x)$. But the Collection of all vertices of the face $G(t, x)$ is identical with $V^*(t, x)$. Therefore, $V^*(t, x) \subset \bar{V}^\circ(t, x)$.

Condition B. There exists a vector $l \in \bar{L}(t, x)$ such that $l'h_i(t) = 0$ for all $i \in I(t, x)$.

Statement 4. Let $I(t, x) \neq \varnothing$ and let Condition B hold. Then the inclusion (3.4) is satisfied.

Indeed, for the vector l from Condition B, the value of $l'C(t)q$ remains constant on $G(t, x)$. Using the inclusion $V^\circ(t, x) \subset G(t, x)$, which follows from Propostion 1, we obtain $\bar{V}^\circ(t, x) = G(t, x)$. Hence, by Statement 1, we obtain the inclusion (3.4).

Statements 1-4 lead to the following theorem.

Theorem 1. If Condition A or Condition B is satisfied in Ω for each point that belongs at least to one SS, then strategy V^* is optimal in Ω.

Let us provide a geometric interpretation of Conditions A and B for the case $n = s = 2$. Condition A implies that the sets $\Pi_1(t)$ and $\Pi_2(t)$ are curves that either do not intersect in $\Omega(t) = \{x \in R^2: (t, x) \in \Omega\}$ (Fig.1a) or their points of intersection are isolated (Fig.1b). Condition B is satisfied, for instance, if the boundary of $W_{\Gamma(t,x)}(t)$ is smooth at the points of the sets $\Pi_1(t)$, $\Pi_2(t)$ (Fig.1c).

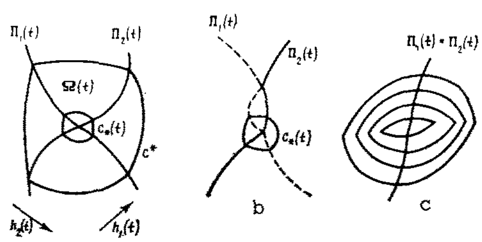

Fig.1

Remarks. 1°. The practical implementation of the strategy V^* is not particularly influenced by violation of Conditions A and B. Indeed, strategy V^* produces an optimal result if there is no sliding mode on the SS.

2°. The method of SS control can be applied also when the set $Q \subset R^s$ can be represented as the sum of segments

$$Q = Q_1 + Q_2 + \ldots + Q_m, \quad Q_i = [-q^i, q^i] \subset R^s$$

Indeed, rewriting the game (1.1) in the form

$$y^\cdot = B(t)u + C(t)Ev, \quad y \in R^n$$
$$u \in P, \quad v \in Q^1 = \{q \in R^m: |q_r| \leqslant 1,\ r = 1, 2, \ldots, m\}$$

where E is a matrix whose rows are the vectors q^i, we obtain the previous case.

4. Example. We will apply the proposed method for constructing the second-player strategy to obtain the worst-case wind disturbance in the problem of controlling the longitudinal motion of an aircraft during landing /1, 5-8/.

The differential equations of the longitudinal motion of the centre of mass of the aircraft linearized with respect to the trim motion along the descent trajectory under the assumption of constant thrust have the form /7/

$$\begin{gathered} x_j^{\cdot} = x_{j+1},\ j = 1, 3, 5;\ x_7^{\cdot} = -4x_7 + 4u \\ x_2^{\cdot} = -0.05x_2 - 0.097x_4 - 0.046x_5 + 0.001x_7 + 0.05x_8 + 0.097x_9 \\ x_4^{\cdot} = 0.241x_2 - 0.639x_4 + 0.79x_5 + 0.026x_7 - 0.241x_8 + 0.639x_9 \\ x_6^{\cdot} = 0.017x_2 + 0.398x_4 - 0.501x_5 - 0.526x_6 - 0.383x_7 - 0.017x_8 - 0.398x_9 \\ \dot{x}_{8,9} = {}^1/_2 x_{8,9} + {}^1/_2 x_{10,11}, \quad \dot{x}_{10,11} = -3x_{10,11} + 3v_{1,2} \\ |u| \leqslant 20,\ |v_1| \leqslant 10,\ |v_2| \leqslant 5 \end{gathered} \tag{4.1}$$

The coordinates x_1, x_3 are the longitudinal and the vertical deviation of the aircraft centre of mass (in meters), and x_5 is the pitching deviation (in degrees). The coordinate x_7 represents the deviation of the elevator (in degrees), and the parameter u is the "specified" deviation of the elevator (in degrees). The coordinates x_8, x_9 represent the deviation of the longitudinal and the vertical components of the wind velocity (m/sec) from their average values. The average values are respectively -5 and 0 and are used in calculating the trim trajectory. The components x_8, x_9 are determined by the generated values of the parameters v_1, v_2. We assume that v_1 and v_2 are the parameters of the second player, whereas u is the parameter of the first player.

Take a fixed time interval $T = [0.15]$. The time $\vartheta = 15c$ is interpreted as the time when the aircraft crosses the edge of the landing strip. We introduce the payoff function $\gamma(x_3, x_4) = \min\{c \geqslant 0: (x_3, x_4) \in cM\}$, where M is a hexagon with the vertices $(-3, 1), (0, 1), (3, 0), (3, -1), (0, -1), (-3, 0)$. We assume that the first player minimizes and the second player maximizes the payoff value γ at time ϑ.

Since the payoff function depends only on two coordinates of the phase vector, we can pass to an equivalent second-order game. The transformation is by a change of variables $y(t) = X(\vartheta, t)x(t)$, where $X(\vartheta, t)$ is the matrix formed from the third and fourth rows of the Cauchy fundamental matrix $\exp[A(\vartheta - t)]$. The equivalent second-order game has the form

$$\begin{gathered} y^{\cdot} = X(\vartheta, t)Bu + X(\vartheta, t)Cv,\ \gamma(y_1, y_2) \\ u \in P = [-20, 20],\ v \in Q = \{(v_1, v_2): |v_1| \leqslant 10,\ |v_2| \leqslant 5\} \end{gathered} \tag{4.2}$$

If $V^*(t, y)$ $(U^*(t, y))$ is the first (second) player optimal strategy in game (4.2), then the strategy $V_*(t, x) = V^*(t, X(\vartheta, t)x)$ $(U_*(t, x) = U^*(t, X(\vartheta, t)x))$ is optimal in game (4.1).

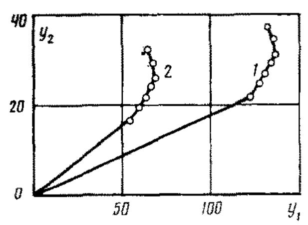

Fig.2

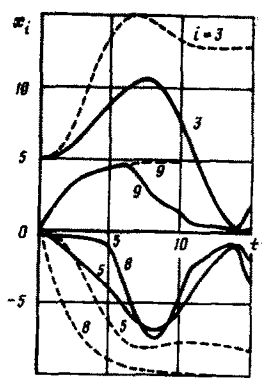

Fig.3

Fix the set $C = \{0.6, 0.8, 1.0, 1.5, 2.0, 4.0, 6.0\}$ of values of the parameter c and the step $\varkappa = 0.05$ between the points t_i that partition the time interval $T = [0, 15]$. For each $c \in C$, the numerically constructed sets $W_c(t_i)$ are convex polyhedra. The value $c = 0.6$ is the least value for which the sets $W_c(t_i)$ are non-empty for all $t_i \in T$.

The strategy V^* (and therefore the strategy V_*) is determined by the two curves $\Pi_1(t), \Pi_2(t)$ that depend on t. The optimal value of the component v_k at time t is determined by the curve $\Pi_k(t)$, $k = 1, 2$. Fig.2 plots the curves $\Pi_1(t)$ and $\Pi_2(t)$ for $t = 5$. (Curves *1* and *2* in Fig.2 respectively are the intersections of the curves $\Pi_1(5), \Pi_2(5)$ with the half-plane $y_2 \geqslant 0$; the parts of $\Pi_1(5), \Pi_2(5)$ that lie below the line $y_2 = 0$ are centrally symmetric about zero to the curves *1*, *2*). The optimal value of $v_1(v_2)$ is -10 (5) to the right of $\Pi_1(5)$ $(\Pi_2(5))$ and 10 (-5) to the left of $\Pi_1(5)$ $(\Pi_2(5))$.

We introduce two first-player controls. The first is the optimal strategy $U_*(t, x) = U^*(t,$

$X(\vartheta, t, x)$. Numerically it is defined by the switching curves /2/. In the second control, the strategy $\bar{U}$ is defined by the formulas

$$a = 0.6x_3(t) + 9x_1(t) + 12x_8(t)$$

$$\bar{U}(t, x) = \begin{cases} a, & |a| \leqslant 20 \\ 20, & a > 20 \\ -20, & a < -20 \end{cases}$$

These relationship roughly model the "linear" law of elevator control used in autopilots /9/.

The step of the discrete scheme for V_*, U_* and $\bar{U}$ is taken equal to 0.05.

Fig.3 plots the variation of the coordinates x_3, x_5, x_8 and x_9 for the initial state $x_{30} = 5$, $x_{j0} = 0$, $j \neq 3$. The value function at the point $(0, x_0)$ is 0.6. The solid curves correspond to the control U_* and the broken curves to $\bar{U}$. The payoff are respectively 0.53 and 5.23. Thus, with optimal behaviour of the first player (in the couple with strategy V_*) the payoff is close to the value of the game in the initial position. If the first player does not follow the optimal strategy, the payoff increases sharply.

Note that there are time intervals from T in which the switching curves $\Pi_1(t)$ and $\Pi_2(t)$ coincide partially or completely and the boundary of the set $W_c(t)$ at the points of their coincidence has a clear break, i.e., Conditions A and B cannot be satisfied. Nevertheless, the modelling results characterize the strategy V_* as optimal in practice.

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